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Bipolar complex fuzzy subalgebras and ideals of BCK/BCI-algebras

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Abstract

The conception of the bipolar complex fuzzy set (BCFS) is one of the fundamental and significant modifications of the fuzzy set (FS) to tackle the tricky and awkward information. BCFS has a rich and wider structure and has been utilized in various fields. In this article, we introduce the concept of bipolar complex fuzzy (BCF) subalgebras (BCFSAs), BCF ideals (BCFIs) of a BCK/BCI-algebra along with certain properties. Further, we investigate the relations between BCFSA and a BCFI and a necessary condition for BCFSA to be a BCFI. We also investigate characterizations of BCFI. Moreover, we introduce the notion of equivalence relations on the collection of all BCFIs of BCK/BCI algebra and the associated properties of equivalence relations.

Keywords: Bipolar complex fuzzy set; BCI/BCK algebra; bipolar complex fuzzy subalgebra/ideal.

1. Introduction

Every time the non-classical logic gets the benefits and rewards of the conventional logic when tackling the data in numerous parts of vagueness and ambiguity. These days non-conventional logic plays a significant role in computer science because it tackles the data containing fuzziness and vagueness. The conception of BCK/BCI-algebras was investigated by Imai and Iseki (1965) and Iseki (1966) which is considered as the modification of propositional logic. Several people studied the concept of BCK/BCI-algebras and employed it in the environment of FS (Akram and Zhan, 2006; Meng, 2000). The structure of FS was explored by Zadeh in 1965 (Zadeh, 1965) in which the membership grade (MG) of elements span the range of $[0, 1]$. The MG shows how much an element is a part of the FS, if the MG is 1, then it implies that the element belongs entirely to the FS and if the MG is 0, then it implies that the element is not contained in the corresponding FS. Mordeson and Malik (1998) investigated fuzzy commutative algebra, and Dubois and Prade (1979) propounded fuzzy real algebra. The $(\epsilon, \epsilon v \mathrm{q})$ and (α, β) fuzzy subalgebras (FSA) in BCI/BCK-algebra was presented by Jun (2005; 2009). FSA of BE-algebras was investigated by Rezaei and Saeid (2011). Akram et al. (2007) explored fuzzy models of $K(G)$ algebras. The fuzzy ideals (FIs) of BCH-algebra were given by Du and Liao (2007). Biswas (1990) investigated fuzzy subgroups and Liu (1982) presented fuzzy invariant subgroups. In the FS, it's hard to indicate the distinction between the contradictory and irrelevant elements and to express the country property of the element. Thus, Zhang (1994) presented the structure of bipolar FS (BFS) in which the positive MG (PMG) of elements span the range of $[0,1]$ and negative MG (NMG) of elements span the range of $[-1,0]$. Akram et al. (2010) explored bipolar fuzzy (BF) K-algebras. Muhiuddin and Al-Kadi (2021) investigated BF implicative ideals in BCK-algebras. Certain characteristics of doubt BF H-ideals in BCI/BCK-algebra were propounded by Al-Masarwah (2018). Muhiuddin (2014) described BF KU-ideals and subalgebras. Muhiuddin et al. (2020) presented novel sort of BF ideals (BFIs) of BCK-algebras. Kawila et al. (2018) explained BF UP-algebras. Lee and Jun (2011) investigated BF a-ideals of BCI-algebras. Jun et al. (2009) explored BF models of certain sorts of ideals in hyper BCK-algebras. Mahmood and Munir (2013) introduced BF subgroups.

There are numerous modifications of the basic conception of FS and complex FS (CFS) is one of them, which was derived by Ramot et al (2022). In CFS, the MG contains both amplitude and phase terms of the elements belonging to $[0,1]$. Afterward, Tamir et al. (2011) presented another form of CFS. Shagagha (2019) investigated complex fuzzy (CF) lie algebras. Jun and Xin (2019 proposed the application of CFSs in BCK/BCI algebra. Rasuli (2022) explored anti CF lie subalgebras. The other and most advanced modification of FS, BFS, and CFS, is BCFS, investigated by Mahmood and Ur Rehman (2022a) in which the PMG of elements belongs to the first quadrant and NMG of elements belongs to the third quadrant of the unit square. The structure of BCFS is a significant tool for tackling tricky and complicated information and generalizing various prevailing concepts. Due to the wide structure and importance of the BCFS, numerous scholars utilized it in various areas (Mahmood et al., 2021; Rehman et al., 2022; Mahmood and Ur Rehman, 2022b). Yang et al. (2022) introduced the conception of BCF subgroups.

Keeping in view the significance and supremacy of the BCFS, in this study we are going to apply the conception of BCFS to the BCK/BCI-algebra. We investigate the conception of BCF subalgebras and BCF ideals of a BCI/BCK-algebra along with associated properties. Moreover, we develop the relation among BCF subalgebra and BCF ideals and a necessary condition for BCFA to be a BCFI. We also investigate characterizations of BCFI. Further, we propound the conception of equivalence relations on the group of al BCF ideals of BCI/BCK-algebra along with associated properties. The underneath article is managed as, In Section 2, we recalled some fundamental notions related to BCK/BCI-algebra and the notion of BCFS. In Section 3, we investigated the concept of BCFSA and BCFI and their related results. The concluding remarks are portrayed in Section 4.

2. Preliminaries

In this Section, we are going to recall some fundamental notions related to BCK/BCI-algebra and the notion of BCFS. The conception of BCI/BCK-algebra is given by Iseki (1966) which has a significant part in the logical algebras and is widely studied by numerous scholars.

Suppose that $\mathfrak{B}(\tau)$ is the collection of all algebras of type $\tau=(2,0)$. BCI-algebra means the set $(\mathcal{Q};*,0)\in\mathfrak{B}(\tau)$ which holds the underneath properties

1. $\Big(\big((\mathrm{q}_1 * \mathrm{q}_2) * (\mathrm{q}_1 * \mathrm{q}_3)\big) * (\mathrm{q}_3 * \mathrm{q}_2) = 0\Big)$ $(\forall\, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3 \in \mathcal{Q})$
2. $\big((\mathrm{q}_1 * (\mathrm{q}_1 * \mathrm{q}_2) * \mathrm{q}_2) = 0\big)$ $(\forall\, \mathrm{q}_1, \mathrm{q}_2 \in \mathcal{Q})$
3. $\big((\mathrm{q}_1 * \mathrm{q}_1) = 0\big)$ $(\forall\, \mathrm{q}_1 \in \mathcal{Q})$
4. $(\mathrm{q}_1 * \mathrm{q}_2 = 0, \mathrm{q}_2 * \mathrm{q}_1 = 0 \Rightarrow \mathrm{q}_1 = \mathrm{q}_2)$ $(\mathrm{q}_1, \mathrm{q}_2 \in \mathcal{Q})$

The BCI-algebra is known as BCK-algebra if it holds the underneath property

5. $(0 * \mathrm{q}_1 = 0)$ $(\forall\, \mathrm{q}_1 \in \mathcal{Q})$

Every BCI/BCK algebra holds the underneath properties

I. $(\mathrm{q}_1 * 0 = 0)$ $(\forall\, \mathrm{q}_1 \in \mathcal{Q})$

II. $(\mathrm{q}_1 \le \mathrm{q}_2 \Rightarrow \mathrm{q}_1 * \mathrm{q}_3 \le \mathrm{q}_2 * \mathrm{q}_3, \mathrm{q}_3 * \mathrm{q}_2 \le \mathrm{q}_3 * \mathrm{q}_1)$ $(\forall\, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3 \in \mathcal{Q})$

III. $\big((\mathrm{q}_1 * \mathrm{q}_2) * \mathrm{q}_3 = (\mathrm{q}_1 * \mathrm{q}_3) * \mathrm{q}_2\big)$ $(\forall\, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3 \in \mathcal{Q})$

IV. $\left((q_1 * q_3) * (q_2 * q_3) \le q_1 * q_2\right) (\forall\, q_1, q_2, q_3 \in \mathcal{Q})$

where, $q_1 \le q_2$ iff $q_1 * q_2 = 0$. Any subset $\mathfrak{D}$ of BCI/BCK-algebra $\mathcal{Q}$ is said to be a subalgebra of BCI/BCK-algebra $\mathcal{Q}$ if $q_1 * q_2 \in \mathfrak{D}\ \forall\, q_1, q_2 \in \mathfrak{D}$. A subset $\mathfrak{I}$ is said to be ideal for BCI/BCK-algebra $\mathcal{Q}$ if it contains 0 and if $q_1 * q_2 \in \mathfrak{I}$ and $q_2 \in \mathfrak{I}$, then $q_1 \in \mathfrak{I}$. The ideal $\mathfrak{I}$ of BCI/BCK-algebra $\mathcal{Q}$ holds the underneath axiom

$$q_1 \le q_2 \text{ and } q_2 \in \mathfrak{I} \Rightarrow q_1 \in \mathfrak{I}$$

An FS Θ in BCI/BCK-algebra $\mathcal{Q}$ is considered a fuzzy subalgebra of $\mathcal{Q}$ if $\forall\, q_1, q_2 \in \mathcal{Q}$

$$\Theta(q_1 * q_2) \ge \min\left(\Theta(q_1), \Theta(q_2)\right)$$

An FS Θ in BCI/BCK-algebra $\mathcal{Q}$ is considered a fuzzy ideal of $\mathcal{Q}$ if $\forall\, q_1, q_2 \in \mathcal{Q}$

$$\Theta(0) \ge \Theta(q_1)$$

$$\Theta(q_1) \ge \min\left(\Theta(q_1 * q_2), \Theta(q_2)\right)$$

The novel structure of BCFS was investigated by Mahmood and Ur Rehman (2022a), given as follow.

The BCFS is of the underneath shape

$$\mathcal{G}_{BCFS} = \left\{\left(q, \left(\Theta^{P}_{\mathcal{G}_{BCFS}}(q), \Theta^{N}_{\mathcal{G}_{BCFS}}(q)\right)\right) \mid q \in \mathcal{Y}\right\} = \left\{\left(q, \begin{pmatrix}\Theta^{RP}_{\mathcal{G}_{BCFS}}(q) + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}(q), \\ \Theta^{RN}_{\mathcal{G}_{BCFS}}(q) + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}(q)\end{pmatrix}\right) \mid q \in \mathcal{Q}\right\}$$

where, $\Theta^{P}_{\mathcal{G}_{BCFS}}(q)$ simplifies the positive membership grade and $\Theta^{N}_{\mathcal{G}_{BCFS}}(q)$ simplifies the negative membership grade and $\Theta^{RP}_{\mathcal{G}_{BCFS}}(q), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q) \in [0,1]$, $\Theta^{RN}_{\mathcal{G}_{BCFS}}(q), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q) \in [-1,0]$. For simplest in this study, we would consider the BCFS as $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$.

3. BCF subalgebras and BCF ideals

Here, we investigate the concept of BCFSA and BCFI and their related results. $\mathcal{Q}$ would be considered as BCK/BCI-algebra in this study unless stated otherwise and $0 = \left(\Theta^{P}_{0_{BCFS}}, \Theta^{N}_{0_{BCFS}}\right) = \left(\Theta^{RP}_{0_{BCFS}} + \iota\, \Theta^{IP}_{0_{BCFS}}, \Theta^{RN}_{0_{BCFS}} + \iota\, \Theta^{IN}_{0_{BCFS}}\right)$ means $0 = (0 + \iota\, 0, -0 - \iota\, 0)$ in this article.

Definition 1: A BCFS $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ in $\mathcal{Q}$ is considered as BCF subalgebra (BCFSA) of $\mathcal{Q}$ if $\forall\, q_1, q_2 \in \mathcal{Q}$, the underneath holds

1. $\Theta^{P}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \ge \min\left(\Theta^{P}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{P}_{\mathcal{G}_{BCFS}}(q_2)\right)$ which means that $\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \ge \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right)$ and $\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \ge \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right)$
2. $\Theta^{N}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \le \max\left(\Theta^{N}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{N}_{\mathcal{G}_{BCFS}}(q_2)\right)$ which means that $\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \le \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right)$ and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_{1,} q_2) \le \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$

Example 1: Suppose a BCK-algebra $\mathcal{Q} = \{0, q_1, q_2, q_3\}$ with Cayley table (Table 1) interpreted as

Table 1. Cayley Table of example 1

*	0	q_1	q_2	q_3
0	0	0	0	0
q_1	q_1	0	0	q_1
q_2	q_2	q_1	0	q_2
q_3	q_3	q_3	q_3	0

Now consider a BCFS $\mathcal{G}_{BCFS}$ in $\mathcal{Q}$ such as

$$\mathcal{G}_{BCFS} = \left\{\left(0, \begin{pmatrix}0,8 + \iota\, 0.5, \\ -0.6 - \iota\, 0.4\end{pmatrix}\right), \left(q_1, \begin{pmatrix}0,8 + \iota\, 0.5, \\ -0.6 - \iota\, 0.4\end{pmatrix}\right), \left(q_2, \begin{pmatrix}0,5 + \iota\, 0.2, \\ -0.3 - \iota\, 0.1\end{pmatrix}\right), \left(q_3, \begin{pmatrix}0,8 + \iota\, 0.5, \\ -0.6 - \iota\, 0.4\end{pmatrix}\right)\right\}$$

Then $\mathcal{G}_{BCFS}$ is a BCFSA of $\mathcal{Q}$.

Example 2: Suppose a BCI-algebra $Q = \{0, \text{q}, \text{q}_1, \text{q}_2, \text{q}_3\}$ with Cayley table (Table 2) interpreted as

Table 2. Cayley Table of example 2

*	0	q	q_1	q_2	q_3
0	0	0	q_3	q_2	q_1
q	q	0	q_3	q_2	q_1
q_1	q_1	q_1	0	q_3	q_2
q_2	q_2	q_2	q_1	0	q_3
q_3	q_3	q_3	q_2	q_1	0

Now consider a BCFS $\mathcal{G}_{BCFS}$ in Q such as

$$\mathcal{G}_{BCFS} = \left\{ \begin{array}{c} \left(0, \begin{pmatrix} 0,9 + \iota\, 0.6, \\ -0.55 - \iota\, 0.5 \end{pmatrix}\right), \left(\text{q}, \begin{pmatrix} 0,7 + \iota\, 0.55, \\ -0.5 - \iota\, 0.45 \end{pmatrix}\right), \left(\text{q}_1, \begin{pmatrix} 0,4 + \iota\, 0.3, \\ -0.2 - \iota\, 0.3 \end{pmatrix}\right), \\ \left(\text{q}_2, \begin{pmatrix} 0,4 + \iota\, 0.3, \\ -0.2 - \iota\, 0.3 \end{pmatrix}\right), \left(\text{q}_3, \begin{pmatrix} 0,4 + \iota\, 0.3, \\ -0.2 - \iota\, 0.3 \end{pmatrix}\right) \end{array} \right\}$$

Then $\mathcal{G}_{BCFS}$ is a BCFSA of Q.

Proposition 1: Suppose a BCFS $\mathcal{G}_{BCFS} = \left(\Theta^P_{\mathcal{G}_{BCFS}}, \Theta^N_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$, if $\mathcal{G}_{BCFS}$ is a BCFSA of Q, then $\forall\, \text{q} \in Q$ $\Theta^P_{\mathcal{G}_{BCFS}}(0) \geq \Theta^P_{\mathcal{G}_{BCFS}}(\text{q})$ which means that $\Theta^{RP}_{\mathcal{G}_{BCFS}}(0) \geq \Theta^{RP}_{\mathcal{G}_{BCFS}}(\text{q})$, $\Theta^{IP}_{\mathcal{G}_{BCFS}}(0) \geq \Theta^{IP}_{\mathcal{G}_{BCFS}}(\text{q})$ and $\Theta^N_{\mathcal{G}_{BCFS}}(0) \leq \Theta^N_{\mathcal{G}_{BCFS}}(\text{q})$ which means that $\Theta^{RN}_{\mathcal{G}_{BCFS}}(0) \leq \Theta^{RN}_{\mathcal{G}_{BCFS}}(\text{q})$, $\Theta^{IN}_{\mathcal{G}_{BCFS}}(0) \leq \Theta^{IN}_{\mathcal{G}_{BCFS}}(\text{q})$.

Proof: Assume that $\text{q} \in Q$, then

$$\Theta^P_{\mathcal{G}_{BCFS}}(0) = \Theta^P_{\mathcal{G}_{BCFS}}(\text{q} * \text{q}) \Rightarrow \Theta^{RP}_{\mathcal{G}_{BCFS}}(0) = \Theta^{RP}_{\mathcal{G}_{BCFS}}(\text{q} * \text{q}) \text{ and } \Theta^{IP}_{\mathcal{G}_{BCFS}}(0) = \Theta^{IP}_{\mathcal{G}_{BCFS}}(\text{q} * \text{q})$$

$$\Rightarrow \Theta^{RP}_{\mathcal{G}_{BCFS}}(0) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\text{q}), \Theta^{RP}_{\mathcal{G}_{BCFS}}(\text{q})\right) = \Theta^{RP}_{\mathcal{G}_{BCFS}}(\text{q}) \text{ and}$$

$$\Theta^{IP}_{\mathcal{G}_{BCFS}}(0) \geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\text{q}), \Theta^{IP}_{\mathcal{G}_{BCFS}}(\text{q})\right) = \Theta^{IP}_{\mathcal{G}_{BCFS}}(\text{q})$$

$$\Rightarrow \Theta^P_{\mathcal{G}_{BCFS}}(0) \geq \Theta^P_{\mathcal{G}_{BCFS}}(\text{q})$$

and

$$\Theta^N_{\mathcal{G}_{BCFS}}(0) = \Theta^N_{\mathcal{G}_{BCFS}}(\text{q} * \text{q}) \Rightarrow \Theta^{RN}_{\mathcal{G}_{BCFS}}(0) = \Theta^{RN}_{\mathcal{G}_{BCFS}}(\text{q} * \text{q}) \text{ and } \Theta^{IN}_{\mathcal{G}_{BCFS}}(0) = \Theta^{IN}_{\mathcal{G}_{BCFS}}(\text{q} * \text{q})$$

$$\Rightarrow \Theta^{RN}_{\mathcal{G}_{BCFS}}(0) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\text{q}), \Theta^{RN}_{\mathcal{G}_{BCFS}}(\text{q})\right) = \Theta^{RN}_{\mathcal{G}_{BCFS}}(\text{q}) \text{ and}$$

$$\Theta^{IN}_{\mathcal{G}_{BCFS}}(0) \leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(\text{q}), \Theta^{IN}_{\mathcal{G}_{BCFS}}(\text{q})\right) = \Theta^{IN}_{\mathcal{G}_{BCFS}}(\text{q})$$

$$\Rightarrow \Theta^N_{\mathcal{G}_{BCFS}}(0) \leq \Theta^N_{\mathcal{G}_{BCFS}}(\text{q}).$$

Definition 2: Consider a BCFS $\mathcal{G}_{BCFS} = \left(\Theta^P_{\mathcal{G}_{BCFS}}, \Theta^N_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$, then

1. The set $\mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\not{f}, \not{g})\right) = \left\{\text{q} \in Q: \Theta^{RP}_{\mathcal{G}_{BCFS}}(\text{q}) \geq \not{f} \text{ and } \Theta^{IP}_{\mathcal{G}_{BCFS}}(\text{q}) \geq \not{g}\right\}$ would be positive $(\not{f}, \not{g})$ −cut of $\mathcal{G}_{BCFS}$, where $\not{f}, \not{g} \in [0, 1]$.
2. The set $\mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right) = \left\{\text{q} \in Q: \Theta^{RN}_{\mathcal{G}_{BCFS}}(\text{q}) \leq d \text{ and } \Theta^{IN}_{\mathcal{G}_{BCFS}}(\text{q}) \leq e\right\}$ would be negative (d, e) −cut of $\mathcal{G}_{BCFS}$, where, $d, e \in [-1, 0]$
3. The set $\mathfrak{PN}\left(\mathcal{G}_{BCFS}, ((\not{f}, \not{g}), (d, e))\right) = \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\not{f}, \not{g})\right) \cap \mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right)$ would be $((\not{f}, \not{g}), (d, e))$ −cut of $\mathcal{G}_{BCFS}$.

Theorem 1: For a BCFSA $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\,\Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\,\Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ of $\mathcal{Q}$, the underneath holds

1. $\forall\, f, g \in [0,1]\ \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right) \neq \emptyset \Rightarrow \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right)$ is a subalgebra of $\mathcal{Q}$.
2. $\forall\, d, e \in [-1,0]\ \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right) \neq \emptyset \Rightarrow \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$ is a subalgebra of $\mathcal{Q}$.

Proof: As $f, g \in [0,1]$ and $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right) \neq \emptyset$. If $q_1, q_2 \in \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right)$, $\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq f$, $\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \geq g$, $\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2) \geq f$, $\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2) \geq g$. Thus

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right) \geq f \text{ and } \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right) \geq g \Rightarrow q_1, * q_2 \in \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right)$$

Hence $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right)$ is a subalgebra of $\mathcal{Q}$.

Now as $d, e \in [-1,0]$ and $\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right) \neq \emptyset$. If $q_1, q_2 \in \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$, $\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1) \leq d$, $\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1) \leq e$, $\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2) \leq d$, $\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2) \leq d$. Thus

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right) \leq d \text{ and } \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right) \leq e \Rightarrow q_1, * q_2 \in \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$$

Hence $\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$ is a subalgebra of $\mathcal{Q}$.

Definition 3: A BCFS $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\,\Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\,\Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ in $\mathcal{Q}$ is considered as BCF ideal (BCFI) of $\mathcal{Q}$ if $\forall\,\forall\, q_1, q_2 \in \mathcal{Q}$, the underneath holds

1. $\Theta^{P}_{\mathcal{G}_{BCFS}}(0) \geq \Theta^{P}_{\mathcal{G}_{BCFS}}(q)$ which means that $\Theta^{RP}_{\mathcal{G}_{BCFS}}(0) \geq \Theta^{RP}_{\mathcal{G}_{BCFS}}(q)$ and $\Theta^{IP}_{\mathcal{G}_{BCFS}}(0) \geq \Theta^{IP}_{\mathcal{G}_{BCFS}}(q)$
2. $\Theta^{N}_{\mathcal{G}_{BCFS}}(0) \leq \Theta^{N}_{\mathcal{G}_{BCFS}}(q)$ which means that $\Theta^{RN}_{\mathcal{G}_{BCFS}}(0) \leq \Theta^{RN}_{\mathcal{G}_{BCFS}}(q)$ and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(0) \leq \Theta^{IN}_{\mathcal{G}_{BCFS}}(q)$
3. $\Theta^{P}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{P}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{P}_{\mathcal{G}_{BCFS}}(q_2)\right)$ which means that $\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right)$ and $\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right)$
4. $\Theta^{N}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{N}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{N}_{\mathcal{G}_{BCFS}}(q_2)\right)$ which means that $\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right)$ and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$

Example 3: Suppose a BCK-algebra $\mathcal{Q} = \{0, q_1, q_2, q_3, q_4\}$ with Cayley table (Table 3) interpreted as

Table 3. Cayley Table of example 3

$*$	0	q_1	q_2	q_3	q_4
0	0	0	0	0	0
q_1	q_1	0	q_1	0	0
q_2	q_2	q_2	0	0	0
q_3	q_3	q_3	q_3	0	0
q_4	q_4	q_3	q_4	q_1	0

Now consider a BCFS $\mathcal{G}_{BCFS}$ in $\mathcal{Q}$ such as

$$\mathcal{G}_{BCFS}=\left\{\begin{matrix}\left(0,\begin{pmatrix}0,67+\iota\,0.5,\\-0.7-\iota\,0.6\end{pmatrix}\right),\left(\mathrm{q}_1,\begin{pmatrix}0,34+\iota\,0.43,\\-0.4-\iota\,0.5\end{pmatrix}\right),\left(\mathrm{q}_2,\begin{pmatrix}0,67+\iota\,0.5,\\-0.7-\iota\,0.6\end{pmatrix}\right),\\\left(\mathrm{q}_3,\begin{pmatrix}0,34+\iota\,0.43,\\-0.4-\iota\,0.5\end{pmatrix}\right),\left(\mathrm{q}_4,\begin{pmatrix}0,34+\iota\,0.43,\\-0.4-\iota\,0.5\end{pmatrix}\right)\end{matrix}\right\}$$

Then $\mathcal{G}_{BCFS}$ is a BCFI of $\mathcal{Q}$.

Proposition 2: Consider a BCFI $\mathcal{G}_{BCFS}=\left(\Theta^{P}_{\mathcal{G}_{BCFS}},\Theta^{N}_{\mathcal{G}_{BCFS}}\right)=\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}+\iota\,\Theta^{IP}_{\mathcal{G}_{BCFS}},\Theta^{RN}_{\mathcal{G}_{BCFS}}+\iota\,\Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ of $\mathcal{Q}$. If in $\mathcal{Q}$, the inequality $\mathrm{q}_1*\mathrm{q}_2\leq\mathrm{q}_3$ holds, then

1. $\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\geq\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right)$ and $\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\geq\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right)$
2. $\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\leq\max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2),\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right)$ and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\leq\max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2),\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right)$

Proof:

1. Assume that $\mathrm{q}_1,\mathrm{q}_2,\mathrm{q}_3\in\mathcal{Q}$ such that $\mathrm{q}_1*\mathrm{q}_2\leq\mathrm{q}_3$, then $(\mathrm{q}_1*\mathrm{q}_2)*\mathrm{q}_3=0$. Thus,

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\geq\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$\geq\min\left(\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}((\mathrm{q}_1*\mathrm{q}_2)*\mathrm{q}_3),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$=\min\left(\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(0),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$=\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right)$$

and

$$\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\geq\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$\geq\min\left(\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}((\mathrm{q}_1*\mathrm{q}_2)*\mathrm{q}_3),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$=\min\left(\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(0),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$=\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right)$$

2. Assume that $\mathrm{q}_1,\mathrm{q}_2,\mathrm{q}_3\in\mathcal{Q}$ such that $\mathrm{q}_1*\mathrm{q}_2\leq\mathrm{q}_3$, then $(\mathrm{q}_1*\mathrm{q}_2)*\mathrm{q}_3=0$. Thus,

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\leq\max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$\leq\max\left(\max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}((\mathrm{q}_1*\mathrm{q}_2)*\mathrm{q}_3),\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right),\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$=\max\left(\max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(0),\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right),\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$=\max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2),\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3)\right)$$

and

$$\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\leq\max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)$$

$$\leq \max\left(\max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}((q_1 * q_2) * q_3), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_3)\right), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

$$= \max\left(\max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(0), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_3)\right), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

$$= \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_3)\right)$$

Proposition 3: Consider a BCFI $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ of $\mathcal{Q}$. If in $\mathcal{Q}$, the inequality $q_1 \leq q_2$ holds, then

1. $\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)$ and $\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \geq \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)$
2. $\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1) \leq \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)$ and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1) \leq \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)$

Proof:

1. Assume that $q_1, q_2, q_3 \in \mathcal{Q}$ such that $q_1 \leq q_2$, then

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$= \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(0), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right) = \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)$$

and

$$\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

$$= \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(0), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right) = \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)$$

2. Assume that $q_1, q_2, q_3 \in \mathcal{Q}$ such that $q_1 \leq q_2$, then

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$= \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(0), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right) = \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)$$

and

$$\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$= \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(0), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right) = \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)$$

Theorem 2: Consider a BCK-algebra $\mathcal{Q}$, in $\mathcal{Q}$, each BCFI of $\mathcal{Q}$ is a BCFSA of $\mathcal{Q}$.

Proof: Assume that $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ is a BCFI of BCK-algebra of $\mathcal{Q}$. As $q_1 * q_2 \leq q_1\ \forall\, q_1, q_2 \in \mathcal{Q}$, thus by proposition (3)

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1),\ \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \text{ and}$$

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \leq \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1),\ \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \leq \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1)$$

and from Def (3), we have

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$\geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$\geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

and

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \leq \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$\leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

$$\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \leq \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$
$$\leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

Therefore, $\mathcal{G}_{BCFS}$ is a BCFSA of $\mathcal{Q}$.

Remark 1: The converse of the above Theorem usually does not hold.

Example 4: Consider example 1 in which $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ is a BCFSA of $\mathcal{Q}$ but $\mathcal{G}_{BCFS}$ is not a BCFI of $\mathcal{Q}$ because

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2) = -0.3 > -0.6 = \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2 * q_1), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1)\right)$$

The following result would show with what condition a BCFSA can be a BCFI in BCK-algebra.

Theorem 3: Suppose a BCFSA $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ of BCK-algebra $\mathcal{Q}$ such that $\forall\, q_1, q_2, q_3 \in \mathcal{Q}$ $\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_3)\right)$ and $\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_3)\right)$, $\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_3)\right)$ and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_3)\right)$ holds and satisfies the inequality $q_1 * q_2 \leq q_3$, then $\mathcal{G}_{BCFS}$ is a BCFI of $\mathcal{Q}$.

Proof: From Proposition (1) we have that $\forall\, q \in \mathcal{Q}$

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(0) \geq \Theta^{RP}_{\mathcal{G}_{BCFS}}(q),\ \Theta^{IP}_{\mathcal{G}_{BCFS}}(0) \geq \Theta^{IP}_{\mathcal{G}_{BCFS}}(q) \text{ and}$$
$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(0) \leq \Theta^{RN}_{\mathcal{G}_{BCFS}}(q),\ \Theta^{IN}_{\mathcal{G}_{BCFS}}(0) \leq \Theta^{IN}_{\mathcal{G}_{BCFS}}(q)$$

As $q_1 * (q_1 * q_2) \leq q_2\ \forall\, q_1, q_2 \in \mathcal{Q}$, so by Proposition (2)

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

and

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_2)\right), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1) \leq \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_2)\right)$$

This implies that $\mathcal{G}_{BCFS}$ is a BCFI of $\mathcal{Q}$.

Theorem 4: Consider a BCFS $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ in $\mathcal{Q}$, then $\mathcal{G}_{BCFS}$ is said to be BCFI of $\mathcal{Q}$ iff the underneath holds

1. $\forall\, f, g \in [0,1]\ \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right) \neq \emptyset \Rightarrow \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right)$ is an ideal of $\mathcal{Q}$.
2. $\forall\, d, e \in [-1,0]\ \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right) \neq \emptyset \Rightarrow \mathfrak{P}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$ is an ideal of $\mathcal{Q}$.

Proof: Let $\mathcal{G}_{BCFS} = \left(\Theta^{P}_{\mathcal{G}_{BCFS}}, \Theta^{N}_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\, \Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ be a BCFI of $\mathcal{Q}$ and $f, g \in [0,1]$, $d, e \in [-1,0]$ such that $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right) \neq \emptyset$ and $\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right) \neq \emptyset$. $0 \in \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right) \cap \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$. Suppose that $q_1, q_2 \in \mathcal{Q}$ such that $q_1 * q_2 \in \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right)$ and $q_2 \in \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (f, g)\right)$ and suppose that $q_3, q_4 \in \mathcal{Q}$ such that $q_3 * q_4 \in \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$ and $q_4 \in \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right)$, then

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq f,\ \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2) \geq f,\ \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_1 * q_2) \geq g,\ \Theta^{IP}_{\mathcal{G}_{BCFS}}(q_2) \geq g \text{ and}$$
$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(q_3 * q_4) \leq d,\ \Theta^{RN}_{\mathcal{G}_{BCFS}}(q_4) \leq d,\ \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_3 * q_4) \leq e,\ \Theta^{IN}_{\mathcal{G}_{BCFS}}(q_4) \leq e$$

By Def (3)

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(q_2)\right) \geq f,$$
$$\Theta^{RI}_{\mathcal{G}_{BCFS}}(q_1) \geq \min\left(\Theta^{RI}_{\mathcal{G}_{BCFS}}(q_1 * q_2), \Theta^{RI}_{\mathcal{G}_{BCFS}}(q_2)\right) \geq g$$

and

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3) \leq \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3 * \mathrm{q}_4), \Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_4)\right) \leq d,$$
$$\Theta^{RI}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3) \leq \max\left(\Theta^{RI}_{\mathcal{G}_{BCFS}}(\mathrm{q}_3 * \mathrm{q}_4), \Theta^{RI}_{\mathcal{G}_{BCFS}}(\mathrm{q}_4)\right) \leq e$$

$\Rightarrow \mathrm{q}_1 \in \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}, \mathscr{g})\right)$ and $\mathrm{q}_3 \in \mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right)$. Thus, $\mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}, \mathscr{g})\right)$ and $\mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right)$ are ideals of $\mathcal{Q}$.

Conversely, assume that conditions 1 and 2 are valid. Suppose that $\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1) = \mathscr{f}$, $\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1) = \mathscr{g}$, $\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1) = d$, and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1) = e$, for any $\mathrm{q}_1 \in \mathcal{Q}$, then $\mathrm{q}_1 \in \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}, \mathscr{g})\right) \cap \mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right) \Rightarrow \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}, \mathscr{g})\right) \neq \emptyset$ and $\mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right) \neq \emptyset$. As $\mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}, \mathscr{g})\right)$ and $\mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right)$ are ideals of $\mathcal{Q}$, $0 \in \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}, \mathscr{g})\right) \cap \mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d, e)\right)$. Thus, $\Theta^{RP}_{\mathcal{G}_{BCFS}}(0) \geq \mathscr{f} = \Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)$, $\Theta^{IP}_{\mathcal{G}_{BCFS}}(0) \geq \mathscr{g} = \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)$, $\Theta^{RN}_{\mathcal{G}_{BCFS}}(0) \leq d = \Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)$ and $\Theta^{IN}_{\mathcal{G}_{BCFS}}(0) \leq e = \Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\ \forall\ \mathrm{q}_1 \in \mathcal{Q}$. If $\exists\ \mathrm{q}'_1, \mathrm{q}'_2, \mathrm{q}'_3, \mathrm{q}'_4 \in \mathcal{Q}$ such that

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1) < \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1 * \mathrm{q}'_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_2)\right),\ \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1) < \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1 * \mathrm{q}'_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_2)\right)$$

and

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3) > \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3 * \mathrm{q}'_4), \Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_4)\right),\ \Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3) > \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3 * \mathrm{q}'_4), \Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_4)\right)$$

then by taking

$$\mathscr{f}_0 = \frac{1}{2}\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1) + \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1 * \mathrm{q}'_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_2)\right)\right)$$
$$\mathscr{g}_0 = \frac{1}{2}\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1) + \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1 * \mathrm{q}'_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_2)\right)\right)$$

and

$$d_0 = \frac{1}{2}\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3) + \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3 * \mathrm{q}'_4), \Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_4)\right)\right)$$
$$e_0 = \frac{1}{2}\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3) + \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3 * \mathrm{q}'_4), \Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_4)\right)\right)$$

We have

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1) < \mathscr{f}_0 < \min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1 * \mathrm{q}'_2), \Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_2)\right)$$
$$\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1) < \mathscr{g}_0 < \min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_1 * \mathrm{q}'_2), \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_2)\right)$$

and

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3) > d_0 > \max\left(\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3 * \mathrm{q}'_4), \Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_4)\right)$$
$$\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3) > e_0 > \max\left(\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_3 * \mathrm{q}'_4), \Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}'_4)\right)$$

Thus, $\mathrm{q}'_1 \notin \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}_0, \mathscr{g}_0)\right)$, $\mathrm{q}'_1 * \mathrm{q}'_2 \in \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}_0, \mathscr{g}_0)\right)$, $\mathrm{q}'_2 \in \mathfrak{P}\left(\Theta^P_{\mathcal{G}_{BCFS}}, (\mathscr{f}_0, \mathscr{g}_0)\right)$, $\mathrm{q}'_3 \notin \mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d_0, e_0)\right)$, $\mathrm{q}'_3 * \mathrm{q}'_4 \in \mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d_0, e_0)\right)$, $\mathrm{q}'_4 \in \mathfrak{N}\left(\Theta^N_{\mathcal{G}_{BCFS}}, (d_0, e_0)\right)$ which is a contradiction and therefore, $\mathcal{G}_{BCFS}$ is BCFI of $\mathcal{Q}$.

Corollary 1: For a BCFI $\mathcal{G}_{BCFS} = \left(\Theta^P_{\mathcal{G}_{BCFS}}, \Theta^N_{\mathcal{G}_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS}} + \iota\,\Theta^{IP}_{\mathcal{G}_{BCFS}}, \Theta^{RN}_{\mathcal{G}_{BCFS}} + \iota\,\Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ of $\mathcal{Q}$, the intersection of a non-empty $(\mathscr{f}, \mathscr{g})$ −cut and a non-empty (d, e) −cut of $\mathcal{G}_{BCFS}$ is an ideal of $\mathcal{Q}$.

Remarks 2:

1. The union of a non-empty $(\mathscr{f}, \mathscr{g})$ −cut and a non-empty (d, e) −cut of $\mathcal{G}_{BCFS}$ may not be an ideal of $\mathcal{Q}$

Example 5: Suppose a BCI-algebra $\mathcal{Q} = \{0, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3\}$ with Cayley table (Table 4) interpreted as

Table 4. Cayley Table of example 5

$*$	0	q_1	q_2	q_3
0	0	q_1	q_2	q_3
q_1	q_1	0	q_3	q_2
q_2	q_2	q_3	0	q_1
q_3	q_3	q_2	q_1	0

Suppose a BCFS in $\mathcal{Q}$ as

$$\mathcal{G}_{BCFS} = \left\{\left(0, \begin{pmatrix} 0,6 + \iota\, 0.4, \\ -0.8 - \iota\, 0.5 \end{pmatrix}\right), \left(q_1, \begin{pmatrix} 0,5 + \iota\, 0.3, \\ -0.4 - \iota\, 0.3 \end{pmatrix}\right), \left(q_2, \begin{pmatrix} 0,4 + \iota\, 0.2, \\ -0.6 - \iota\, 0.4 \end{pmatrix}\right), \left(q_3, \begin{pmatrix} 0,4 + \iota\, 0.2, \\ -0.3 - \iota\, 0.1 \end{pmatrix}\right)\right\}$$

Then,

$$\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (\not{f}, \not{g})\right) = \begin{cases} \emptyset & if\ 0.6 < \not{f} \le 1, and\ 0.4 < \not{g} \le 1 \\ \{0\} & if\ 0.5 < \not{f} \le 0.6, and\ 0.3 < \not{g} \le 0.4 \\ \{0, q_1\} & if\ 0.4 < \not{f} \le 0.5, and\ 0.2 < \not{g} \le 0.3 \\ \mathcal{Q} & if\ 0 \le \not{f} \le 0.4, and\ 0 \le \not{g} \le 0.2, \end{cases}$$

and

$$\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (d, e)\right) = \begin{cases} \emptyset & if -1 \le d < -0.8, and -1 \le e < -0.5 \\ \{0\} & if -0.8 \le d < -0.6, and -0.5 \le e < -0.4 \\ \{0, q_2\} & if -0.6 \le d < -0.4, and -0.4 \le e < -0.1 \\ \mathcal{Q} & if -0.4 \le d < 0, and -0.1 \le e < 0, \end{cases}$$

It is clear from Theorem (4) that $\mathcal{G}_{BCFS}$ is a BCFI of $\mathcal{Q}$. But $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (0.5, 0.4)\right) \cup \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (-0.6, -0.4)\right) = \{0, q_1\} \cup \{0, q_2\} = \{0, q_1, q_2\}$ and not a BCFI of $\mathcal{Q}$ because $q_3 * q_1 = q_2 \in \{0, q_1, q_2\}$ but $q_3 \notin \{0, q_1, q_2\}$.

2. If $d = -\not{f}$ and $e = -\not{g}$, then also the union of a non-empty $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (\not{f}, \not{g})\right)$ and a non-empty $\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}}, (-\not{f}, -\not{g})\right)$ of $\mathcal{G}_{BCFS}$ is not an ideal of $\mathcal{Q}$.

Example 6 Suppose a BCI-algebra $\mathcal{Q} = \{0, 1, q_1, q_2, q_3\}$ with Cayley table (Table 5) interpreted as

Table 5. Cayley Table of example 6

$*$	0	1	q_1	q_2	q_3
0	0	0	q_1	q_2	q_3
1	1	0	q_1	q_2	q_3
q_1	q_1	q_1	0	q_3	q_2
q_2	q_2	q_2	q_3	0	q_1
q_3	q_3	q_3	q_2	q_1	0

Suppose a BCFS in $\mathcal{Q}$ as

$$\mathcal{G}_{BCFS} = \left\{ \begin{matrix} \left(0, \begin{pmatrix} 0,9 + \iota\, 0.6, \\ -0.6 - \iota\, 0.5 \end{pmatrix}\right), \left(1, \begin{pmatrix} 0,7 + \iota\, 0.4, \\ -0.6 - \iota\, 0.5 \end{pmatrix}\right), \left(q_1, \begin{pmatrix} 0,4 + \iota\, 0.3, \\ -0.3 - \iota\, 0.2 \end{pmatrix}\right), \\ \left(q_2, \begin{pmatrix} 0,2 + \iota\, 0.1, \\ -0.4 - \iota\, 0.3 \end{pmatrix}\right), \left(q_3, \begin{pmatrix} 0,2 + \iota\, 0.1, \\ -0.3 - \iota\, 0.2 \end{pmatrix}\right) \end{matrix} \right\}$$

Then,

$$\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}}, (\not{f}, \not{g})\right) = \begin{cases} \emptyset & if\ 0.9 < \not{f} \le 1, and\ 0.6 < \not{g} \le 1 \\ \{0\} & if\ 0.7 < \not{f} \le 0.9, and\ 0.4 < \not{g} \le 0.6 \\ \{0, 1\} & if\ 0.4 < \not{f} \le 0.7, and\ 0.3 < \not{g} \le 0.4 \\ \{0,1, q_1\} & if\ 0.2 < \not{f} \le 0.4, and\ 0.1 < \not{g} \le 0.3 \\ \mathcal{Q} & if\ 0 \le \not{f} \le 0.2, and\ 0 \le \not{g} \le 0.1, \end{cases}$$

and

$$\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(d,e)\right)=\begin{cases}\emptyset & if-1\le d<-0.6, and-1\le e<-0.5\\ \{0,1\} & if-0.6\le d<-0.4, and-0.5\le e<-0.3\\ \{0,1,\mathrm{q}_2\} & if-0.4\le d<-0.3, and-0.3\le e<-0.2\\ \mathcal{Q} & if-0.3\le d<0, and-0.2\le e<0,\end{cases}$$

It is clear from Theorem (4) that $\mathcal{G}_{BCFS}$ is a BCFI of $\mathcal{Q}$. But $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(0.4,0.3)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-0.4,-0.3)\right)=\{0,1,\mathrm{q}_1\}\cup\{0,1,\mathrm{q}_2\}=\{0,1,\mathrm{q}_1,\mathrm{q}_2\}$ and not a BCFI of $\mathcal{Q}$ because $\mathrm{q}_3*\mathrm{q}_1=\mathrm{q}_2\in\{0,1,\mathrm{q}_1,\mathrm{q}_2\}$ but $\mathrm{q}_3\notin\{0,\mathrm{q}_1,\mathrm{q}_2\}$.

Theorem 5: If $\mathcal{G}_{BCFS}=\left(\Theta^{P}_{\mathcal{G}_{BCFS}},\Theta^{N}_{\mathcal{G}_{BCFS}}\right)=\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}+\iota\,\Theta^{IP}_{\mathcal{G}_{BCFS}},\Theta^{RN}_{\mathcal{G}_{BCFS}}+\iota\,\Theta^{IN}_{\mathcal{G}_{BCFS}}\right)$ is a BCFI of $\mathcal{Q}$ such that $\forall\ \mathrm{q}\in\mathcal{Q}$,

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q})+\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q})\ge 0 \text{ and } \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q})+\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q})\ge 0 \qquad (1)$$

then, the union of a non-empty (f,g) −cut and a non-empty $(-f,-g)$ −cut (i.e. $d=-f,e=-g$) of $\mathcal{G}_{BCFS}$ is an ideal of $\mathcal{Q}$.

Proof: Suppose $f,g\in[0,1]$ and as $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\neq\emptyset$ and $\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)\neq\emptyset$, are the ideal of $\mathcal{Q}$ by employing Theorem (4). Thus, $0\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cap\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$. Suppose that $\mathrm{q}_1,\mathrm{q}_2\in\mathcal{Q}$ such that $\mathrm{q}_1*\mathrm{q}_2\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$ and $\mathrm{q}_2\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$. From this, we get the four cases

i. $\mathrm{q}_1*\mathrm{q}_2\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)$ and $\mathrm{q}_2\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)$

ii. $\mathrm{q}_1*\mathrm{q}_2\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)$ and $\mathrm{q}_2\in\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$

iii. $\mathrm{q}_1*\mathrm{q}_2\in\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$ and $\mathrm{q}_2\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)$

iv. $\mathrm{q}_1*\mathrm{q}_2\in\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$ and $\mathrm{q}_2\in\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$

(i) implies that $\mathrm{q}_1\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\subseteq\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$, (iv) implies that $\mathrm{q}_1\in\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)\subseteq\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$. For (ii), we have

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2)\ge f,\ \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2)\ge g \text{ and } \Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\le -f,\ \Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\le -g$$

So from Def (3) and Eq. (1), we have

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\ge\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),-\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge f$$
$$\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\ge\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),-\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge g$$
$$\Rightarrow\mathrm{q}_1\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\subseteq\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$$

now (iii) implies that

$$\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2)\le -f,\ \Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2)\le -g \text{ and } \Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\ge f,\ \Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\ge g$$

So from Def (3) and Eq. (1), we have

$$\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\ge\min\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge\min\left(-\Theta^{RN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{RP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge f$$
$$\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1)\ge\min\left(\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge\min\left(-\Theta^{IN}_{\mathcal{G}_{BCFS}}(\mathrm{q}_1*\mathrm{q}_2),\Theta^{IP}_{\mathcal{G}_{BCFS}}(\mathrm{q}_2)\right)\ge g$$

$\Rightarrow\mathrm{q}_1\in\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\subseteq\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$. Thus, $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS}},(f,g)\right)\cup\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(-f,-g)\right)$ is an ideal of $\mathcal{Q}$.

Suppose that $BCFI(\mathcal{Q})$ is the group of all BCFIs of $\mathcal{Q}$, $f,g\in[0,1]$ and $d,e\in[-1,0]$, then we introduce binary relations $\mathfrak{R}P^{f,g}$ and $\mathfrak{R}N^{d,e}$ on $BCFI(\mathcal{Q})$ as

$$(\mathcal{G}_{BCFS-1},\mathcal{G}_{BCFS-2})\in\Re P^{f,g}\Leftrightarrow\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS-1}},(f,g)\right)=\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS-2}},(f,g)\right)$$
$$(\mathcal{G}_{BCFS-1},\mathcal{G}_{BCFS-2})\in\Re N^{d,e}\Leftrightarrow\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS-1}},(d,e)\right)=\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS-2}},(d,e)\right)$$

respectively, $\forall\ \mathcal{G}_{BCFS-1}=\left(\Theta^{P}_{\mathcal{G}_{BCFS-1}},\Theta^{N}_{\mathcal{G}_{BCFS-1}}\right)=\left(\Theta^{RP}_{\mathcal{G}_{BCFS-1}}+\iota\,\Theta^{IP}_{\mathcal{G}_{BCFS-1}},\Theta^{RN}_{\mathcal{G}_{BCFS-1}}+\iota\,\Theta^{IN}_{\mathcal{G}_{BCFS-1}}\right)$ and $\mathcal{G}_{BCFS-2}=\left(\Theta^{P}_{\mathcal{G}_{BCFS-2}},\Theta^{N}_{\mathcal{G}_{BCFS-2}}\right)=\left(\Theta^{RP}_{\mathcal{G}_{BCFS-2}}+\iota\,\Theta^{IP}_{\mathcal{G}_{BCFS-2}},\Theta^{RN}_{\mathcal{G}_{BCFS-2}}+\iota\,\Theta^{IN}_{\mathcal{G}_{BCFS-2}}\right)\in BCFI(\mathcal{Q})$. Obviously, $\Re P^{f,g}$ and $\Re N^{d,e}$ are equivalence relations on $BCFI(\mathcal{Q})$. For $\mathcal{G}_{BCFS-1}=\left(\Theta^{P}_{\mathcal{G}_{BCFS-1}},\Theta^{N}_{\mathcal{G}_{BCFS-1}}\right)=\left(\Theta^{RP}_{\mathcal{G}_{BCFS-1}}+\iota\,\Theta^{IP}_{\mathcal{G}_{BCFS-1}},\Theta^{RN}_{\mathcal{G}_{BCFS-1}}+\iota\,\Theta^{IN}_{\mathcal{G}_{BCFS-1}}\right)\in BCFI(\mathcal{Q})$, suppose that $[\mathcal{G}_{BCFS-1}]_{\Re P^{f,g}}\left([\mathcal{G}_{BCFS-1}]_{\Re N^{d,e}}\right)$ is an equivalence class of $\mathcal{G}_{BCFS-1}$ modular $\Re P^{f,g}\left(\Re N^{d,e}\right)$. Further, we would signify the collection of all equivalence classes modular $\Re P^{f,g}\left(\Re N^{d,e}\right)$ by $BCFI(\mathcal{Q})/\Re P^{f,g}\left(BCFI(\mathcal{Q})/\Re N^{d,e}\right)$ i.e.

$$BCFI(\mathcal{Q})/\Re P^{f,g}=\left\{[\mathcal{G}_{BCFS-1}]_{\Re P^{f,g}}\mid\mathcal{G}_{BCFS-1}\in BCFI(\mathcal{Q})\right\}$$
$$BCFI(\mathcal{Q})/\Re N^{d,e}=\left\{[\mathcal{G}_{BCFS-1}]_{\Re N^{d,e}}\mid\mathcal{G}_{BCFS-1}\in BCFI(\mathcal{Q})\right\}$$

Next, Suppose that the collection of all ideals of $\mathcal{Q}$ is signified by $Ideal(\mathcal{Q})$, define maps

$$\Gamma_{(f,g)}:BCFI(\mathcal{Q})\rightarrow Ideal(\mathcal{Q})\cup\{\emptyset\}\text{ by }\Gamma_{(f,g)}(\mathcal{G}_{BCFS})=\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS-1}},(f,g)\right)$$

and

$$\Gamma_{(d,e)}:BCFI(\mathcal{Q})\rightarrow Ideal(\mathcal{Q})\cup\{\emptyset\}\text{ by }\Gamma_{(d,e)}(\mathcal{G}_{BCFS})=\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS-2}},(d,e)\right)$$

$\forall\ \mathcal{G}_{BCFS}=\left(\Theta^{P}_{\mathcal{G}_{BCFS}},\Theta^{N}_{\mathcal{G}_{BCFS}}\right)=\left(\Theta^{RP}_{\mathcal{G}_{BCFS}}+\iota\,\Theta^{IP}_{\mathcal{G}_{BCFS}},\Theta^{RN}_{\mathcal{G}_{BCFS}}+\iota\,\Theta^{IN}_{\mathcal{G}_{BCFS}}\right)\in BCFI(\mathcal{Q})$. Cleary $\Gamma_{(f,g)}$ and $\Gamma_{(d,e)}$ are well defined.

Theorem 6: Let $f,g\in(0,1]$ and $d,e\in[-1,0)$, then maps $\Gamma_{(f,g)}$ and $\Gamma_{(d,e)}$ are surjective.

Proof: Cleary a BCFS $0=\left(\Theta^{P}_{0_{BCFS}},\Theta^{N}_{0_{BCFS}}\right)=\left(\Theta^{RP}_{0_{BCFS}}+\iota\,\Theta^{IP}_{0_{BCFS}},\Theta^{RN}_{0_{BCFS}}+\iota\,\Theta^{IN}_{0_{BCFS}}\right)$ is a BCFI of $\mathcal{Q}$, where $\forall$ $\mathrm{q}\in 0$, $\Theta^{RP}_{0_{BCFS}}(\mathrm{q})=\Theta^{IP}_{0_{BCFS}}(\mathrm{q})=\Theta^{RN}_{0_{BCFS}}(\mathrm{q})=\Theta^{IN}_{0_{BCFS}}(\mathrm{q})=0$. Then

$$\Gamma_{(f,g)}(0)=\mathfrak{P}\left(\Theta^{P}_{0_{BCFS}},(f,g)\right)=\emptyset$$

and

$$\Gamma_{(d,e)}(0)=\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS}},(d,e)\right)=\emptyset$$

now for any non-empty $\mathcal{H}\in Ideal(\mathcal{Q})$, suppose a BCFS $\mathcal{G}(\mathcal{H})_{BCFS}=\left(\Theta^{P}_{\mathcal{G}(\mathcal{H})_{BCFS}},\Theta^{N}_{\mathcal{G}(\mathcal{H})_{BCFS}}\right)=\left(\Theta^{RP}_{\mathcal{G}(\mathcal{H})_{BCFS}}+\iota\,\Theta^{IP}_{\mathcal{G}(\mathcal{H})_{BCFS}},\Theta^{RN}_{\mathcal{G}(\mathcal{H})_{BCFS}}+\iota\,\Theta^{IN}_{\mathcal{G}(\mathcal{H})_{BCFS}}\right)$ in $\mathcal{Q}$, and

$$\Theta^{P}_{\mathcal{G}(\mathcal{H})_{BCFS}}:\mathcal{Q}\rightarrow[0,1]+\iota\,[0,1]\text{ defined as }\Theta^{P}_{\mathcal{G}(\mathcal{H})_{BCFS}}=\begin{cases}1+\iota\,1 & if\ \mathrm{q}\in\mathcal{H}\\ 0+\iota\,0 & otherwise\end{cases}$$

and

$$\Theta^{N}_{\mathcal{G}(\mathcal{H})_{BCFS}}:\mathcal{Q}\rightarrow[-1,0]+\iota\,[-1,0]\text{ defined as }\Theta^{N}_{\mathcal{G}(\mathcal{H})_{BCFS}}=\begin{cases}-1-\iota\,1 & if\ \mathrm{q}\in\mathcal{H}\\ -0-\iota\,0 & otherwise\end{cases}$$

Then clearly $\mathcal{G}(\mathcal{H})$ is a BCFI of $\mathcal{Q}$. Next, we have

$$\Gamma_{(f,g)}\left(\mathcal{G}(\mathcal{H})\right)=\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}(\mathcal{H})_{BCFS}},(f,g)\right)=\left\{\mathrm{q}\in\mathcal{Q}\mid\Theta^{RP}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})\geq f\ \&\ \Theta^{IP}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})\geq g\right\}$$
$$=\left\{\mathrm{q}\in\mathcal{Q}\mid\Theta^{RP}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})=1\ \&\ \Theta^{IP}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})=1\right\}=\mathcal{H}$$

and

$$\Gamma_{(d,e)}\left(\mathcal{G}(\mathcal{H})\right)=\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}(\mathcal{H})_{BCFS}},(d,e)\right)=\left\{\mathrm{q}\in\mathcal{Q}\mid\Theta^{RN}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})\leq d\ \&\ \Theta^{IN}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})\leq e\right\}$$
$$\left\{\mathrm{q}\in\mathcal{Q}\mid\Theta^{RN}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})=-1\ \&\ \Theta^{IN}_{\mathcal{G}(\mathcal{H})_{BCFS}}(\mathrm{q})=-1\right\}=\mathcal{H}$$

Thus, $\Gamma_{(f,g)}$ and $\Gamma_{(d,e)}$ are onto (surjective).

Theorem 7: For every $f,g\in(0,1]$ and $d,e\in[-1,0)$, the quotient sets $BCFI(\mathcal{Q})/\Re P^{f,g}$ and $BCFI(\mathcal{Q})/\Re N^{d,e}$ are equipotent to $Ideal(\mathcal{Q})\cap\{\emptyset\}$.

Proof: Define two maps

$$\Gamma'_{(\mathscr{f},\mathscr{g})}: BCFI(\mathcal{Q})/\Re P^{\mathscr{f},\mathscr{g}} \rightarrow Ideal(\mathcal{Q}) \cup \{\emptyset\} \text{ as } \Gamma'_{(\mathscr{f},\mathscr{g})}\left([\mathcal{G}_{BCFS}]_{\Re P^{\mathscr{f},\mathscr{g}}}\right) = \Gamma_{(\mathscr{f},\mathscr{g})}(\mathcal{G}_{BCFS})$$

and

$$\Gamma'_{(d,e)}: BCFI(\mathcal{Q})/\Re N^{d,e} \rightarrow Ideal(\mathcal{Q}) \cup \{\emptyset\} \text{ as } \Gamma'_{(d,e)}\left([\mathcal{G}_{BCFS}]_{\Re N^{d,e}}\right) = \Gamma_{(d,e)}(\mathcal{G}_{BCFS})$$

For $\mathscr{f},\mathscr{g} \in (0,1]$ and $d,e \in [-1,0)$ and $\forall\, \mathcal{G}_{BCFS} \in BCFI(\mathcal{Q})$. Now each $\mathcal{G}_{BCFS-1} = \left(\Theta^{P}_{\mathcal{G}_{BCFS-1}}, \Theta^{N}_{\mathcal{G}_{BCFS-1}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS-1}} + \iota\,\Theta^{IP}_{\mathcal{G}_{BCFS-1}}, \Theta^{RN}_{\mathcal{G}_{BCFS-1}} + \iota\,\Theta^{IN}_{\mathcal{G}_{BCFS-1}}\right)$ and $\mathcal{G}_{BCFS-2} = \left(\Theta^{P}_{\mathcal{G}_{BCFS-2}}, \Theta^{N}_{\mathcal{G}_{BCFS-2}}\right) = \left(\Theta^{RP}_{\mathcal{G}_{BCFS-2}} + \iota\,\Theta^{IP}_{\mathcal{G}_{BCFS-2}}, \Theta^{RN}_{\mathcal{G}_{BCFS-2}} + \iota\,\Theta^{IN}_{\mathcal{G}_{BCFS-2}}\right) \in BCFI(\mathcal{Q})$, if $\mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS-1}}, (\mathscr{f},\mathscr{g})\right) = \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}_{BCFS-2}}, (\mathscr{f},\mathscr{g})\right)$ and $\mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS-1}}, (d,e)\right) = \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}_{BCFS-2}}, (d,e)\right)$, then $(\mathcal{G}_{BCFS-1}, \mathcal{G}_{BCFS-2}) \in \Re P^{\mathscr{f},\mathscr{g}}$ and $(\mathcal{G}_{BCFS-1}, \mathcal{G}_{BCFS-2}) \in \Re N^{d,e}$, thus $[\mathcal{G}_{BCFS-1}]_{\Re P^{\mathscr{f},\mathscr{g}}} = [\mathcal{G}_{BCFS-2}]_{\Re P^{\mathscr{f},\mathscr{g}}}$ and $[\mathcal{G}_{BCFS-1}]_{\Re N^{d,e}} = [\mathcal{G}_{BCFS-2}]_{\Re N^{d,e}}$. Consequently, $\Gamma'_{(\mathscr{f},\mathscr{g})}$ and $\Gamma'_{(d,e)}$ are one-one (injective). For non-empty $\mathcal{H} \in Ideal(\mathcal{Q})$, suppose a BCFS $\mathcal{G}(\mathcal{H})_{BCFS} = \left(\Theta^{P}_{\mathcal{G}(\mathcal{H})_{BCFS}}, \Theta^{N}_{\mathcal{G}(\mathcal{H})_{BCFS}}\right) = \left(\Theta^{RP}_{\mathcal{G}(\mathcal{H})_{BCFS}} + \iota\,\Theta^{IP}_{\mathcal{G}(\mathcal{H})_{BCFS}}, \Theta^{RN}_{\mathcal{G}(\mathcal{H})_{BCFS}} + \iota\,\Theta^{IN}_{\mathcal{G}(\mathcal{H})_{BCFS}}\right)$ in $\mathcal{Q}$ presented in Theorem (6), then

$$\Gamma'_{(\mathscr{f},\mathscr{g})}\left([\mathcal{G}(\mathcal{H})_{BCFS}]_{\Re P^{\mathscr{f},\mathscr{g}}}\right) = \Gamma_{(\mathscr{f},\mathscr{g})}(\mathcal{G}(\mathcal{H})_{BCFS}) = \mathfrak{P}\left(\Theta^{P}_{\mathcal{G}(\mathcal{H})_{BCFS}}, (\mathscr{f},\mathscr{g})\right) = \mathcal{H}$$

and

$$\Gamma'_{(d,e)}\left([\mathcal{G}(\mathcal{H})_{BCFS}]_{\Re N^{d,e}}\right) = \Gamma_{(d,e)}(\mathcal{G}(\mathcal{H})_{BCFS}) = \mathfrak{N}\left(\Theta^{N}_{\mathcal{G}(\mathcal{H})_{BCFS}}, (d,e)\right) = \mathcal{H}$$

Next, For BCFI $0 = \left(\Theta^{P}_{0_{BCFS}}, \Theta^{N}_{0_{BCFS}}\right) = \left(\Theta^{RP}_{0_{BCFS}} + \iota\,\Theta^{IP}_{0_{BCFS}}, \Theta^{RN}_{0_{BCFS}} + \iota\,\Theta^{IN}_{0_{BCFS}}\right)$, we have

$$\Gamma'_{(\mathscr{f},\mathscr{g})}\left([0_{BCFS}]_{\Re P^{\mathscr{f},\mathscr{g}}}\right) = \Gamma_{(\mathscr{f},\mathscr{g})}(0_{BCFS}) = \mathfrak{P}\left(\Theta^{P}_{0_{BCFS}}, (\mathscr{f},\mathscr{g})\right) = \emptyset$$

and

$$\Gamma'_{(d,e)}\left([0_{BCFS}]_{\Re N^{d,e}}\right) = \Gamma_{(d,e)}(0_{BCFS}) = \mathfrak{N}\left(\Theta^{N}_{0_{BCFS}}, (d,e)\right) = \emptyset$$

Hence, $\Gamma'_{(\mathscr{f},\mathscr{g})}$ and $\Gamma'_{(d,e)}$ are surjective (onto) and the required proof is completed.

4. Conclusion

We developed this study by keeping in view the importance and significance of the BCFS theory as BCFS theory is one of the richest and most modified theories in the prevailing literature. This article contained the conception of BCFSAs and BCFIs of BCK/BCI-algebras with various properties. Also, this article contained the relations between BCFSA and a BCFI and a significant condition for BCFSA to be a BCFI. Furthermore, we proposed characterizations of BCFI in this study. At last, this study contained the conception of equivalence relations on the group of all BCFIs of BCK/BCI-algebra and the linked properties.

In the future, we would like to employ the investigated work to some of the prevailing works like BCF soft sets (Mahmood et al, 2022), bipolar complex intuitionistic FSs (Al-Husban, 2022; Jan, 2022).

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